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On fuzzy spin spaces†

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Abstract. The operational meaning of fuzzy measurement of two spin components S_x and S_y is examined. Spectral densities assigning probabilities in each spin state to fuzzy simultaneous values of S_x and S_y are introduced, and their informational completeness is examined.

1. Introduction

It has been shown recently that after extending the framework of probability theory to spaces of fuzzy sample points, it becomes possible to express the quantum mechanical state of a system of spinless particles as a probability distribution on fuzzy phase space (Prugovečki 1976a, b, Ali and Prugovečki 1977). This leads to a formulation of quantum statistical mechanics which bears a remarkable resemblance to classical statistical mechanics (Prugovečki 1976c).

In this paper, we shall examine how this framework might be extended to include spin. We limit our considerations to spin-$\frac{1}{2}$, which displays all the essential features of the general case.

To see how fuzzy sample points do result from spin measurements, let us consider the prototype of all such measurements, namely the Stern–Gerlach experiment performed on an atom $\mathscr{A}$. To measure the spin component S_x of $\mathscr{A}$ when $\mathscr{A}$ travels along the z axis, the Stern–Gerlach set-up correlates the S_x value of $\mathscr{A}$ to the position $\boldsymbol{R}$ of its centre of mass by passing $\mathscr{A}$ through a magnetic field $\boldsymbol{H}$ whose H_x component has a non-zero gradient pointing in the direction of the positive x axis, while the other two components, in principle, could be made constant. If at time $t=0$, prior to its passing through the magnetic field, $\mathscr{A}$ was in the state

$$\Psi_0 = u(\boldsymbol{R};0)\sum_{\xi=-1/2}^{+1/2}\gamma_\xi\Phi_\xi(\boldsymbol{r}), \qquad \sum_\xi|\gamma_\xi|^2=1, \tag{1.1}$$

where $\Phi_{-1/2}$ and $\Phi_{+1/2}$ are mutually orthogonal internal states corresponding to the respective values $-\frac{1}{2}$ and $+\frac{1}{2}$ of S_x, then upon passing through the field, the state of $\mathscr{A}$ at time t is (Gottfried 1966, § 19):

$$\Psi_t = \sum_{\xi=-1/2}^{+1/2}\gamma_\xi u_\xi(\boldsymbol{R};t)\Phi_\xi(\boldsymbol{r}). \tag{1.2}$$

Thus, if $\mathbb{R}^{(x)}_{+1/2}$ and $\mathbb{R}^{(x)}_{-1/2}$ denote the half-spaces of $\mathbb{R}^3$ above and below the (y, z) plane,

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respectively, and if at time t we detect $\mathscr{A}$ in $\mathbb{R}_\xi^{(x)}$, the probability that the value of its spin component S_x was μ equals

$$
\begin{aligned}
|\gamma_\mu|^2 &\approx \int_{\mathbb{R}_\mu^{(x)}} \mathrm{d}\boldsymbol{R} \int \mathrm{d}\boldsymbol{r} |\Psi_t(\boldsymbol{R}, \boldsymbol{r})|^2 = |\gamma_\mu|^2 \chi'_\mu(\mu), \\
\chi'_\xi(\mu) &= \int_{\mathbb{R}_\xi^{(x)}} |u_\mu(\boldsymbol{R}; t)|^2 \, \mathrm{d}\boldsymbol{R}, \qquad \xi, \mu = -\tfrac{1}{2}, +\tfrac{1}{2}.
\end{aligned}
\tag{1.3}
$$

In practice, by choosing t sufficiently large, and the apparatus sufficiently massive, one can achieve $\chi'_\xi(\xi) \approx 1$ to a degree of accuracy which can be identified with *practical* certainty. However, it is important to recall that as a consequence of the conservation of angular momentum, for any given apparatus $\chi'_\xi(\xi)$ has an upper limit *strictly smaller* than one regardless of how large t is chosen (Wigner 1952, Araki and Yanase 1960, Park 1968). That upper limit can be raised only by increasing the size of the apparatus, and therefore the case $\chi'_\xi(\xi) = 1$ is an asymptotic limit requiring an apparatus of infinite size for its realization.

In accordance with the general definition of a fuzzy sample point (Prugovečki 1976a), the pair (ξ, χ'_ξ) constitutes a fuzzy value for S_x. The value $\chi'_\xi(\mu)$ of the confidence function χ'_ξ clearly represents a measure of the certainty that when a *reading* ξ is obtained the actual value of S_x was μ.

Using the spin eigenstates of S_x,

$$
S_x \psi'_\mu = \mu \psi'_\mu, \qquad \mu = -\tfrac{1}{2}, +\tfrac{1}{2}, \qquad \langle \psi'_\mu | \psi'_{\mu'} \rangle = \delta_{\mu\mu'}, \tag{1.4}
$$

we can introduce a *spectral density*

$$
F_\xi^{S_x} = \sum_{\mu=-1/2}^{+1/2} |\psi'_\mu\rangle \chi'_\xi(\mu) \langle \psi'_\mu|, \qquad \xi = -\tfrac{1}{2}, +\tfrac{1}{2}, \tag{1.5}
$$

in spin space. Its expectation value

$$
P_\psi^{S_x}(\xi) = \langle \psi | F_\xi^{S_x} \psi \rangle = \sum_\mu \chi'_\xi(\mu) |\langle \psi'_\mu | \psi \rangle|^2 \tag{1.6}
$$

for an arbitrary spin state ψ equals the probability that a measurement of S_x would yield the fuzzy value (ξ, χ'_ξ). Obviously, the conventional case of perfectly sharp measurements is recovered when $\chi'_\xi(\mu) = \delta_{\xi\mu}$.

Let us imagine now that at the same time t we measure also the component S_y of spin, obtaining for $\eta = -\frac{1}{2}, +\frac{1}{2}$ the fuzzy value (η, χ''_η). This simultaneous measurement of S_x and S_y results in values in the sample space

$$
\hat{\mathscr{S}}_{x,y} = \{(\xi, \chi'_\xi) \times (\eta, \chi''_\eta) | \xi, \eta = \pm\tfrac{1}{2}\} \tag{1.7}
$$

consisting of four distinct fuzzy values for the pair (S_x, S_y) of spin observables. For example, in the Stern–Gerlach experiment, this could be achieved by arranging that not only H_x but also H_y possess a non-zero gradient in the positive direction of their respective axes. However, in this particular measurement set-up, nothing substantially new is thus achieved since the net result of the two gradients would be a gradient in some new direction $\boldsymbol{n}$ in between the x axis and the y axis. Yet, the general question arises whether there are probability distributions $P_\psi^{S_x, S_y}(\xi, \eta)$ on $\hat{\mathscr{S}}_{x,y}$ which have the correct marginal values (1.6) for S_x, as well as the corresponding correct marginal values for S_y (cf (2.8)–(2.9)).

If the sample space $\hat{\mathscr{S}}_{x,y}$ consists of sharp sample points, i.e. if $\chi'_\xi(\mu)=\delta_{\xi\mu}$ and $\chi''_\eta(\nu)=\delta_{\eta\nu}$, then it is already known that the answer is negative (Margenau and Hill 1961). In the next section we shall derive necessary and sufficient conditions which the confidence functions $\chi'_\xi(\mu)$ and $\chi''_\eta(\nu)$ have to satisfy in order to guarantee the existence of such probability distributions. These conditions confirm the result of Margenau and Hill for spaces of sharp sample points, but at the same time they reveal a whole class of spaces of fuzzy sample points for which the probability distributions satisfying the required marginality conditions exist for all spin states ψ. Consequently, we are able to show in § 3 that the simultaneous measurement of S_x and S_y can be used, in principle, to pinpoint an arbitrary mixed state in spin space, and not just the eigenstates of S_x or S_y. Thus, in this respect the situation is very much the same as with simultaneous measurements of position and momentum, although there are also substantial differences resulting from the very different nature of the spin spectra on the one hand, and the spectra of position and momentum observables on the other.

2. Spectral densities on the fuzzy spin space $\hat{\mathscr{S}}_{x,y}$

In defining $\hat{\mathscr{S}}_{x,y}$ we have required that all the confidence functions are normalized

$$\sum_\mu \chi'_\xi(\mu)=1, \qquad \sum_\nu \chi''_\eta(\nu)=1 \tag{2.1}$$

as well as spectrum-normalized (Prugovečki 1976a)

$$\sum_\xi \chi'_\xi(\mu)=1, \qquad \sum_\eta \chi''_\eta(\nu)=1. \tag{2.2}$$

Hence, if the probability of simultaneously measuring the fuzzy values (ξ,χ'_ξ) and (η,χ''_η) for S_x and S_y, respectively, is to be expressed in terms of a spectral density $F_{\xi\eta}$,

$$P_\psi^{S_x,S_y}(\xi,\eta)=\langle\psi|F_{\xi\eta}\psi\rangle, \tag{2.3}$$

that density has to satisfy the marginality conditions (Prugovečki 1976a)

$$\sum_\eta F_{\xi\eta}=F_\xi^{S_x}, \tag{2.4}$$

$$\sum_\xi F_{\xi\eta}=F_\eta^{S_y}, \tag{2.5}$$

where, by analogy with (1.5),

$$F_\eta^{S_y}=\sum_\nu |\psi''_\nu\rangle\chi''_\eta(\nu)\langle\psi''_\nu|, \tag{2.6}$$

$$S_y\psi''_\nu=\nu\psi''_\nu, \qquad \nu=-\tfrac{1}{2},+\tfrac{1}{2}, \qquad \langle\psi''_\nu|\psi''_{\nu'}\rangle=\delta_{\nu\nu'}. \tag{2.7}$$

The reasons for imposing these conditions lie in the essential requirements that

$$\sum_\eta P_\psi^{S_x,S_y}(\xi,\eta)=P_\psi^{S_x}(\xi), \tag{2.8}$$

$$\sum_\xi P_\psi^{S_x,S_y}(\xi,\eta)=P_\psi^{S_y}(\eta), \tag{2.9}$$

for arbitrary spin states $\psi\in\mathscr{F}$.

The question now arises whether there exist in the spin Hilbert space $\mathscr{F}$ positive-definite operators

$$F_{\xi\eta} \geqslant 0 \tag{2.10}$$

that satisfy (2.4) and (2.5) in a given fuzzy spin space $\hat{\mathscr{S}}_{x,y}$.

For spin-$\frac{1}{2}$, all confidence functions in $\hat{\mathscr{S}}_{x,y}$ can be specified in terms of two parameters p' and p'':

$$\chi'_+(+\tfrac{1}{2}) = 1-\chi'_+(-\tfrac{1}{2}) = 1-\chi'_-(+\tfrac{1}{2}) = \chi'_-(-\tfrac{1}{2}) = p', \tag{2.11}$$

$$\chi''_+(+\tfrac{1}{2}) = 1-\chi''_+(-\tfrac{1}{2}) = 1-\chi''_-(+\tfrac{1}{2}) = \chi''_-(-\tfrac{1}{2}) = p''. \tag{2.12}$$

This is a consequence of (2.1) and (2.2). (Here, as well as in the following, we used the abbreviations $\pm$ for the subscripts $\pm\frac{1}{2}$.)

Let us set

$$\langle\psi'_+|F_{\xi\eta}\psi'_+\rangle = a_{\xi\eta}, \tag{2.13}$$

$$\langle\psi'_-|F_{\xi\eta}\psi'_-\rangle = b_{\xi\eta}, \tag{2.14}$$

$$\langle\psi'_+|F_{\xi\eta}\psi'_-\rangle = \langle\psi'_-|F_{\xi\eta}\psi'_+\rangle^* = c_{\xi\eta}. \tag{2.15}$$

The marginality conditions (2.4) are equivalent to

$$\sum_\eta \langle\psi'_\mu|F_{\xi\eta}\psi'_\nu\rangle = \chi'_\xi(\mu)\delta_{\mu\nu}, \tag{2.16}$$

and consequently they are satisfied if and only if

$$a_{++}+a_{+-} = b_{-+}+b_{--} = p', \tag{2.17}$$

$$a_{-+}+a_{--} = b_{++}+b_{+-} = 1-p', \tag{2.18}$$

$$c_{++}+c_{+-} = c_{-+}+c_{--} = 0. \tag{2.19}$$

After taking into consideration that

$$\psi''_\pm = 2^{-1/2}(\psi'_+ \pm \psi'_-), \tag{2.20}$$

and using the Hermiticity of $F_{\xi\eta}$, we easily arrive at

$$\langle\psi''_\pm|F_{\xi\eta}\psi''_\pm\rangle = \tfrac{1}{2}(a_{\xi\eta}+b_{\xi\eta}) \pm \operatorname{Re} c_{\xi\eta}, \tag{2.21}$$

$$\langle\psi''_+|F_{\xi\eta}\psi''_-\rangle = \tfrac{1}{2}(a_{\xi\eta}-b_{\xi\eta}) - \mathrm{i}\operatorname{Im} c_{\xi\eta}. \tag{2.22}$$

Now we employ the marginality condition (2.5). This condition is equivalent to the relation

$$\sum_\xi \langle\psi''_\mu|F_{\xi\eta}\psi''_\nu\rangle = \chi''_\eta(\nu)\delta_{\mu\nu}, \tag{2.23}$$

which by some straightforward algebra is seen to be in its turn equivalent to the following set of equations:

$$a_{++}+a_{-+}+b_{++}+b_{-+} = 1, \tag{2.24}$$

$$a_{++}+a_{-+} = b_{++}+b_{-+}, \tag{2.25}$$

$$a_{+-}+a_{--}+b_{+-}+b_{--} = 1, \tag{2.26}$$

$$a_{+-}+a_{--} = b_{+-}+b_{--}, \tag{2.27}$$

$$\mathrm{Re}(c_{++}+c_{-+}) = -\mathrm{Re}(c_{+-}+c_{--}) = p''-\tfrac{1}{2}, \tag{2.28}$$

$$\mathrm{Im}(c_{++}+c_{-+}) = \mathrm{Im}(c_{+-}+c_{--}) = 0. \tag{2.29}$$

The Hermiticity of $F_{\xi\eta}$ requires that $a_{\xi\eta}$ and $b_{\xi\eta}$ be real. Thus (2.17)–(2.19) and (2.24)–(2.29) represents a system of sixteen linear algebraic equations for the sixteen real quantities $a_{\xi\eta}$, $b_{\xi\eta}$, Re $c_{\xi\eta}$ and Im $c_{\xi\eta}$, ξ, $\eta = \pm\frac{1}{2}$. These equations are not, however, independent, with the result that the values of p' and p'' and of the quantities a_{--}, b_{--} and c_{--} can be chosen arbitrarily. All the remaining quantities can then be expressed in terms of those chosen values:

$$a_{++} = a_{--}+p'-\tfrac{1}{2}, \tag{2.30}$$

$$b_{++} = b_{--}+\tfrac{1}{2}-p', \tag{2.31}$$

$$c_{++} = c_{--}+p''-\tfrac{1}{2}, \tag{2.32}$$

$$a_{\pm\mp} = \tfrac{1}{2}-a_{\mp\mp}; \qquad b_{\pm\mp} = \tfrac{1}{2}-b_{\mp\mp}; \qquad c_{\pm\mp} = -c_{\pm\pm}. \tag{2.33}$$

Up to this point we have taken advantage of the Hermiticity of $F_{\xi\eta}$, but not of its positive-definiteness property (2.10). For (2.10) to be true, it is necessary and sufficient that the roots $\lambda_{1,2}$ of

$$(a_{\xi\xi}-\lambda)(b_{\xi\eta}-\lambda) = |c_{\xi\eta}|^2 \tag{2.34}$$

be non-negative, which in turn is true if and only if the inequalities

$$a_{\xi\eta}+b_{\xi\eta} \geq 0, \tag{2.35}$$

$$|c_{\xi\eta}|^2 \leq a_{\xi\eta}b_{\xi\eta}, \tag{2.36}$$

hold for all ξ, $\eta = \pm\frac{1}{2}$.

It is easy to derive from (2.30)–(2.33) that (2.35) is satisfied if and only if

$$0 \leq a_{--}+b_{--} \leq 1. \tag{2.37}$$

On the other hand, the four inequalities represented in (2.36) give rise to the following necessary and sufficient conditions on $a = a_{--}$, $b = b_{--}$, $c = c_{--}$ and p', p'':

$$|c|^2 \leq \min\{ab, (1-p'-a)(p'-b)\}, \tag{2.38}$$

$$(c+p''-\tfrac{1}{2})^2 \leq \min\{(a+p'-\tfrac{1}{2})(b+\tfrac{1}{2}-p'), (\tfrac{1}{2}-a)(\tfrac{1}{2}-b)\}. \tag{2.39}$$

These conditions, in conjunction with (2.37), imply that

$$\max\{0, \tfrac{1}{2}-p'\} \leq a \leq \min\{\tfrac{1}{2}, 1-p'\}, \tag{2.40}$$

$$\max\{0, p'-\tfrac{1}{2}\} \leq b \leq \min\{\tfrac{1}{2}, p'\}. \tag{2.41}$$

The above inequalities (2.38) to (2.41) represent a set of necessary and sufficient conditions for the existence of a spectral density $F_{\xi\eta}$: to every choice of non-negative a, b, complex c and $0 \leq p', p'' \leq 1$ that satisfies these four inequalities corresponds one such spectral density computable from (2.30)–(2.33).

It is evident that, in general, there is an infinity of spectral densities $F_{\xi\eta} \geq 0$ satisfying the marginality conditions (2.4) and (2.5) for various choices of confidence functions χ'_η and χ''_η in accordance with (2.11) and (2.12), but that such solutions do not exist for arbitrary choices of p' and p'' since (2.38) and (2.39) impose restrictions on the admissible values of p'' for given p'. In particular, it is easily seen that there are no solutions for the case of simultaneous sharp values of S_x and S_y, i.e., when $p' \in \{0, 1\}$,

$p'' \in \{0, 1\}$. This is to be expected in the light of the results of Margenau and Hill (1961). In fact, when $p' = 1$, we obtain $a = c = 0$ from (2.38), and then deduce $p'' = \frac{1}{2}$ by using (2.39), while for $p' = 0$ we obtain $b = c = 0$ from (2.38) and again $p'' = \frac{1}{2}$. Because of the obviously symmetric roles played by S_x and S_y in the problem, we infer that $p'' \in \{0, 1\}$ implies $p' = \frac{1}{2}$. Thus, a perfectly accurate measurement of S_x is compatible only with complete uncertainty in our information on the values of S_y, and vice versa if the measured values of S_y are sharp.

3. Informational completeness for spin observables

Generally speaking, a family of observables is said to be *informationally* complete with respect to a given quantum mechanical state if the probability distribution for those observables when the system is in the aforementioned state specifies that state uniquely (Prugovečki 1976d). In the context of spin observables, each spin component $S_{\boldsymbol{n}}$ along some direction $\boldsymbol{n}$ is informationally complete with respect to states represented by the eigenvectors of $S_{\boldsymbol{n}}$ (e.g., S_x is informationally complete for spin states represented by ψ'_ξ, $\xi = -\frac{1}{2}, +\frac{1}{2}$, and S_y with respect to ψ''_η, $\eta = -\frac{1}{2}, +\frac{1}{2}$). Yet, $S_{\boldsymbol{n}}$ is not informationally complete *globally*, i.e., with respect to all spin states. On the other hand, given any pure spin state represented by some $\psi \in \mathscr{F}$, it follows from the irreducibility of the representation of SU(2) whose infinitesimal generators are S_x, S_y and S_z that there is some direction $\boldsymbol{n}$ for which ψ is an eigenvector of $S_{\boldsymbol{n}}$. Thus, the family of *all* spin projections $S_{\boldsymbol{n}}$ certainly is informationally complete.

The last observation is not, however, as useful as it might seem at first glance from the point of view of unambigously determining a given spin state ψ by measuring $S_{\boldsymbol{n}}$, since in order to use it for that purpose one would have to know the orientation of $\boldsymbol{n}$ prior to measurement. Hence it is of interest to examine whether the simultaneous (fuzzy) measurement of two spin components, say S_x and S_y, might lead to global informational completeness. Rephrased more precisely, the question posed is whether there are spectral densities $F_{\xi\eta}$ on $\hat{\mathscr{S}}_{x,y}$ such that the equalities

$$\mathrm{Tr}(F_{\xi\eta}\rho_1) = \mathrm{Tr}(F_{\xi\eta}\rho_2), \qquad \xi, \eta = -\tfrac{1}{2}, +\tfrac{1}{2}, \tag{3.1}$$

for any two density matrices ρ_1 and ρ_2 in $\mathscr{F}$ imply that $\rho_1 = \rho_2$.

In the preceding section we have computed all spectral densities $F_{\xi\eta}$ satisfying the required positivity and marginality conditions for the case of spin-$\frac{1}{2}$. After introducing $\alpha = \rho_1 - \rho_2$, and rewriting (3.1) in the form

$$a_{\xi\eta}\alpha_{++} + c_{\xi\eta}\alpha_{+-} + c^*_{\xi\eta}\alpha_{-+} + b_{\xi\eta}\alpha_{--} = \mathrm{Tr}(F_{\xi\eta}\alpha) = 0, \tag{3.2}$$

we see that the question of the existence of an informationally complete spectral density $F_{\xi\eta}$ reduces to whether the determinant of the system of the four linear equations for $\alpha_{++}, \ldots, \alpha_{--}$ that are represented in (3.2) is zero or not.

It is easily seen that among spectral densities computed in the preceding section there are many for which that determinant is zero, e.g., all those for which $c = c_{--}$, and therefore also c_{++}, c_{+-} and c_{-+}, are real. We note that in particular this will be the case when the measurements of either S_x or S_y are perfectly accurate, so that either $p' = 1$ or $p'' = 1$. Furthermore, consider also the relation

$$\langle\psi|F_{\xi\eta}\psi\rangle = \int_{\mathbb{R}^{(x)}_\xi \cap \mathbb{R}^{(y)}_\eta} \mathrm{d}\boldsymbol{R} \int \mathrm{d}\boldsymbol{r}|\Psi_t(\boldsymbol{R}, \boldsymbol{r})|^2, \qquad \psi = \begin{pmatrix}\gamma_+\\ \gamma_-\end{pmatrix}, \tag{3.3}$$

which defines the spectral density for the Stern–Gerlach experiment treated as in § 1 to accommodate the simultaneous measurement of S_x and S_y. It is easy to see that even thus re-interpreted, the Stern–Gerlach experiment does not supply any data on ψ beyond the absolute values of γ_+ and γ_-. Hence the corresponding matrices $F_{\xi\eta}$ do not provide a non-vanishing determinant for (3.2).

Yet, there do exist systems of 2×2 matrices $F_{\xi\eta}$ satisfying all the conditions of § 3 for which the determinant in question does not vanish. One example of such a system is obtained when $p'=\frac{3}{4}$, $p''=\frac{5}{8}$, $a=\frac{1}{8}$, $b=\frac{3}{8}$ and $c=\frac{1}{8}\mathrm{i}$.

Thus, we conclude that the existence of spectral densities $F_{\xi\eta}$ that guarantee the global informational completeness of $\{S_x, S_y\}$ is consistent with the spin formalism.

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